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QUALITATIVE BEHAVIOUR OF SOLUTIONS OF THE GOURSAT PROBLEM FOR HYPERBOLIC DIFFERENTIAL EQUATIONS

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Abstract—Oscillation criteria and asymptotic behavior of solutions of characteristic initial-value problems have been extended to the Goursat problem.

1. INTRODUCTION

The characteristic initial-value problem

$$(\mathrm{P1}) \begin{cases} u_{xy} + p(x, y)u = 0, \\ u(x, 0) = \phi(x), \\ u(0, y) = \psi(y) \end{cases}$$

has been studied in $R^+ \times R^+$ by several authors with the aim of establishing oscillation criteria and determining the asymptotic behavior of solutions. The works of Kreith [1–3], Pagan [4, 5], Yoshida [6], Kreith and Swanson [7] and Georgiou and Kreith [8], established some techniques for handling such problems. The present paper is an attempt to extend some of these results to the Goursat problem:

$$u_{xy} + p(x, y)u = 0, \quad x > 0, 0 < y < g(x), \tag{1}$$

$$\left.\begin{aligned} u(x, 0) &= \phi(x), \quad x \geqslant 0, \\ u(x, g(x)) &= \psi(x), \quad x \geqslant 0; \end{aligned}\right\} \tag{2}$$

subject to the compatibility condition $\phi(0) = \psi(0)$. Here g is strictly increasing continuous function defined for $x \geqslant 0$, satisfying $g(0) = 0$. Functions ϕ and ψ are continuously differentiable and p is continuous in

$$D = \{(x, y): x \geqslant 0, \quad 0 \leqslant y \leqslant g(x)\}.$$

A solution of problem (1, 2) is said to be oscillatory if it vanishes along a curve (nodal line) in the xy-plane. The existence and uniqueness of solutions of problem (1, 2) have been established in Refs [9,10, pp. 147–178].

In Section 2 we will establish criteria under which the Goursat problem (1, 2) has an oscillatory solution.

Section 3 deals with nonoscillatory solutions of problem (1, 2) for $p \leqslant 0$ and $p \geqslant 0$.

In section 4 we study the asymptotic behavior of the solution of problem (1, 2) using techniques similar to those of Kreith and Swanson [7].

2. OSCILLATION CRITERIA FOR THE GOURSAT PROBLEM

Pagan [4] has shown that the solution of the characteristic initial-value problem (P1) changes sign in $R^+ \times R^+$ if the conditions

(i) $\quad p(x, y) \geqslant K^2 > 0,$

(ii) $\quad \phi(x) > 0, \quad \psi(y) > 0,$

and

(iii) $\phi'(x) \leqslant 0, \quad \psi'(y) \leqslant 0,$

in $R^+ \times R^+$ are satisfied. The fact that under such conditions we should not expect oscillatory solutions for problem (1, 2) can be seen by example $u(x, y) = e^{y-g(x)}$, which is the solution of

$$u_{xy} + g'(x)u = 0,$$
$$u(x, 0) = e^{-g(x)},$$
$$u(x, g(x)) = 1,$$

even though $p(x, y) = g'(x)$ can be taken arbitrarily large and positive.

The following result establishes the fact that the solution of problem (1, 2) can be oscillatory, regardless of the size of p, as long as p is nonnegative.

Theorem 2.1

Let u satisfy problem (1, 2). If

(i) $p(x, y) \geqslant 0,$

(ii) $\phi(x) > 0, \quad \psi(x) > 0,$

(iii) $\displaystyle\int^{\infty} [g(x)\phi'(x) + g'(x)\psi(x)]\,dx = -\infty,$

then u changes sign in D.

Proof. Define $U(x) = \displaystyle\int_0^{g(x)} u(x, y)\,dy$, then

$$U'(x) = \int_0^{g(x)} u_x(x, y)\,dy + g'(x)u(x, g(x)). \tag{3}$$

From equation (1), we have

$$u_x(x, y) = u_x(x, 0) - \int_0^y p(x, \eta)u(x, \eta)\,d\eta.$$

Substituting the above in equation (3) yields

$$U'(x) = g(x)\phi'(x) + g'(x)\psi(x) - \int_0^{g(x)} \int_0^y p(x, \eta)u(x, \eta)\,d\eta\,dy. \tag{4}$$

Now suppose that u is of fixed sign in D. Without loss of generality, we assume it is positive. Then from hypothesis (i) and equality (4), we have

$$U(x) \leqslant U(M) + \int_M^x [g(s)\phi'(s) + g'(s)\psi(s)]\,ds.$$

By hypothesis (iii), as x approaches infinity the r.h.s. of the above inequality becomes negative. This is a contradiction because u was assumed positive in D.

Example 2.1

Let

$$g(x) = x, \quad \phi(x) = \frac{1}{x+1} \quad \text{and} \quad \psi(x) = \frac{1}{(x+1)^2}.$$

Then

$$\int_M^x \left[\frac{-s}{(s+1)^2} + \frac{1}{(s+1)^2}\right] ds = -\ln(x+1) - \frac{2}{x+1} + \ln(M+1) + \frac{2}{M+1},$$

which approaches $-\infty$ as $x \to \infty$.

Note. For $p \equiv 0$ the data of the above example yields the solution

$$u(x, y) = -\frac{1}{y+1} + \frac{1}{(y+1)^2} + \frac{1}{x+1},$$

which changes sign along the hyperbola $(y+1)^2 - (y+1)(x+1) + (x+1) = 0$ in the region $x > 0,\ 0 < y < x$.

The above method of proof applies equally well to the nonlinear Goursat problem:

$$u_{xy} + f(x, y, u) = 0, \quad (x, y) \text{ in } D, \tag{5}$$

$$\left.\begin{aligned} u(x, 0) &= \phi(x), \quad x \geqslant 0,\\ u(x, g(x)) &= \psi(x), \quad x \geqslant 0; \end{aligned}\right\} \tag{6}$$

where $f(x, y, \xi) \geqslant 0$, for $\xi \geqslant 0$, and $f(x, y, -\xi) = -f(x, y, \xi)$.
This fact is noted as follows.

Corollary 2.1

Let u satisfy the nonlinear Goursat problem (5, 6). If

(i) $\phi(x) > 0, \quad \psi(x) > 0,$

and

(ii) $$\int^{\infty} [g(x)\phi'(x) + g'(x)\psi(x)]\,\mathrm{d}x = -\infty,$$

then u changes sign in D.

In the preceding theorems we have sought criteria which assert that u cannot stay positive $\forall(x, y)$ in D. A variant of this property is the following: $u(x, y)$ cannot satisfy $u > \epsilon\ \forall(x, y)$ in D, even though $u(x, 0)$ and $u(x, g))$ are $>\epsilon$, $\forall x \geqslant 0$. In this regard we have the following theorem.

Theorem 2.2

Let $\epsilon > 0$ be a given constant. If

(i) $p(x, y) \geqslant 0, \quad (x, y)$ in D,

(ii) $\phi(x) > \epsilon, \quad \psi(x) > \epsilon, \quad x \geqslant 0,$

and

(iii) $$\int^{\infty} \left\{ g(s)\phi'(s) + [\psi(s) - \epsilon]g'(s) - \epsilon \int_0^{g(s)} [g(s) - y]p(s, y)\,\mathrm{d}y \right\} \mathrm{d}s = -\infty,$$

then the solution of the Goursat problem (1, 2) cannot satisfy $u(x, y) > \epsilon\ \forall(x, y)$ in D.

Proof. Let $v = u - \epsilon$, so that problem (1, 2) becomes

$$v_{xy} + (v + \epsilon)p(x, y) = 0, \tag{7}$$

$$\left.\begin{aligned} v(x, 0) &= \phi(x) - \epsilon,\\ v(x, g(x)) &= \psi(x) - \epsilon. \end{aligned}\right\}$$

Define the function v by

$$v(x) = \int_0^{g(x)} v(x, y)\,\mathrm{d}y.$$

Differentiating v with respect to x yields

$$v'(x) = g'(x)v(x, g(x)) + \int_0^{g(x)} v_x(x, y)\,\mathrm{d}y. \tag{8}$$

From equation (7), we have

$$v_x(x, y) = v_x(x, 0) - \int_0^y p(x, \eta)[v(x, \eta) + \epsilon]\,\mathrm{d}\eta.$$

Substituting this value of $v_x(x, y)$ into equation (8), we obtain

$$V'(x) = \phi'(x)g(x) + g'(x)[\psi(x) - \epsilon] - \int_0^{g(x)} \int_0^{y} p(x, \eta)\ [v(x, \eta) + \epsilon]\, d\eta\, dy$$

which upon changing the order of integration, yields

$$\begin{aligned} V'(x) &= \phi'(x)g(x) + g'(x)[\psi(x) - \epsilon] - \int_0^{g(x)} \int_\eta^{g(x)} p(x, \eta)[v(x, \eta) + \epsilon]\, dy\, d\eta \\ &= \phi'(x)g(x) + g'(x)[\psi(x) - \epsilon] - \int_0^{g(x)} [g(x) - y]p(x, y)[v(x, y) + \epsilon]\, dy. \end{aligned}$$

Now assume to the contrary that $u > \epsilon\ \forall\ (x, y)$ in D. Then $v > 0$, and by hypothesis (i) we have

$$V'(x) \leqslant \phi'(x)g(x) + g'(x)[\psi(x) - \epsilon] - \epsilon \int_0^{g(x)} [g(x) - y]p(x, y)\, dy.$$

Integrating both sides of the above inequality on $[M, x]$, we have

$$V(x) \leqslant V(M) + \int_M^x \left\{ \phi'(s)g(s) + g'(s)[\psi(s) - \epsilon] - \epsilon \int_0^{g(s)} [g(s) - y]p(s, y)\, dy \right\} ds.$$

If we let $x \to \infty$ in the above inequality, the r.h.s. becomes negative due to assumption (iii). This contradiction establishes the theorem.

The following corollary, which follows from Theorem 2.2, is very similar to Pagen's [4] result, mentioned at the beginning of Section 2.

Corollary 2.2

Let u satisfy the Goursat problem (1, 2). If

(i) $\lim_{x\to\infty} g(x) < \infty$,

(ii) $p(x, y) \geqslant K^2 > 0$, K const,

(iii) $\phi(x) > \epsilon$, $\psi(x) > \epsilon$,

and

(iv) $\phi'(x) \leqslant 0$, $\psi'(x) \leqslant 0$,

then u cannot satisfy $u(x, y) > \epsilon\ \forall (x, y)$ in D.

Proof. Since $\psi'(x) \leqslant 0$ there exists x_1 and L, such that $\psi(x) \leqslant L$, for $x \geqslant x_1$. Therefore,

$$\begin{aligned} \lim_{x\to\infty} \int_M^x g'(x)\psi(s)\, ds &\leqslant \lim_{x\to\infty} \left[\int_M^{x_1} g'(s)\psi(s)\, ds + L \int_{x_1}^x g'(s)\, ds \right] \\ &= \int_M^{x_1} g'(s)\psi(s)\, ds + L\left(\lim_{x\to\infty} [g(x) - g(x_1)] \right) < \infty. \end{aligned} \tag{9}$$

Also, since g is increasing there exist x_2 and N, such that $g(x) \geqslant N$, for $x \geqslant x_2$. Therefore,

$$\lim_{x\to\infty} \int_M^x g^2(s)\, ds \geqslant \lim_{x\to\infty} \left[\int_M^{x_2} g^2(s)\, ds + \int_{x_2}^x N^2\, ds \right] = \infty. \tag{10}$$

From inequalities (9) and (10) and assumption $\phi'(x) \leqslant 0$, we have

$$\begin{aligned} &\lim_{x\to\infty} \int^x \left\{ g(s)\phi'(s) + (\psi - \epsilon)g'(s) - \epsilon \int_0^{g(s)} [g(s) - y]K^2\, dy \right\} ds \\ &\qquad = \lim_{x\to\infty} \int^x \left[g(s)\phi'(s) + (\psi - \epsilon)g'(s) - \frac{\epsilon K^2}{2} g^2(s) \right] ds = -\infty. \end{aligned} \tag{11}$$

Equation (11) and assumptions (ii) and (iii) imply that the conditions of Theorem 2.2 are satisfied. Therefore, by Theorem 2.2, u cannot satisfy $u(x, y) > \epsilon\ \forall (x, y)$ in D.

The following theorem establishes an oscillation criterion for the Goursat problem that requires no extra restriction on g, and the size of $P \geqslant 0$ is still irrelevant.

Theorem 2.3

Let u satisfy the Goursat problem (1, 2). If

(i) $p(x, y) \geqslant 0$,

(ii) $\phi(x) > 0, \quad \psi(x) > 0$,

(iii) $\lim_{x \to \infty} \phi(x) = 0$

and

(iv) there exists a point $x_1 > 0$, such that $\psi(x_1) < \phi(x_1)$, then u changes sign in D.

Proof. Let (x, y) be a point in D such that $y = g(x_1)$. Then for such y, the value of u at (x, y) is given by

$$u(x, y) = u(g^{-1}(y), y) - u(g^{-1}(y), 0) + u(x, 0) - \int_{g^{-1}(y)}^{x} \int_{0}^{y} P(\xi, \eta) u(\xi, \eta) \, d\eta \, d\xi. \tag{12}$$

Suppose u is positive in D. Then from equation (12) and assumption (i), we have

$$u(x, y) \leqslant u(g^{-1}(y), y) - u(g^{-1}(y), 0) + u(x, 0) = \psi(x_1) - \phi(x_1) + \phi(x).$$

Letting $x \to \infty$ in the above inequality we reach a contradiction.

An analogous argument applies if u is negative in D.

Example 2.2

Let $P \equiv 0$, then for $\phi(x) = e^{-x}$, $\psi(x) = e^{-2x}$ and $g(x) = x$, u will be given by $u(x, y) = e^{-x} + e^{-2y} - e^{-y}$. The nodal line of the solution u is the graph of the equation $e^{-x} + e^{-2y} - e^{-y} = 0$, $0 < y < x$, or

$$x = -\ln \, (e^{-y} - e^{-2y}). \tag{13}$$

We note that this curve represents a single-valued function of y, but is multivalued as a function of x. The distance between the point $(-\ln(e^{-y} - e^{-2y}), y)$ on the curve. (13), and the point (y, y) on the line $y = x$ is given by

$$d(y) = |y + \ln(e^{-y} - e^{-2y})|.$$

Since

$$\lim_{y \to \infty} d(y) = 0,$$

the line $y = x$ is an asymptote of equation (13). The lower branch of equation (13),

$$y = -\ln\left(\frac{1 + \sqrt{1 - 4e^{-x}}}{2}\right), \quad x \geqslant 2 \ln 2,$$

is asymptotic to $y = 0$.

The question one would now like to ask is whether under the conditions of Theorem 2.3 and $p > 0$ the nodal lines of the solution of problem (1, 2) always represent functions of y only? If so, are they asymptotic to the curve $y = g(x)$ and the x-axis, as in Example 2.2?

In the case of the characteristic initial-value problem Pagan [4] used the implicit function theorem to show that in the first quadrant, the first nodal line is qualitatively similar to a hyperbola. He also showed that the first nodal line is asymptotic to the x- and y-axes.

For the Goursat problem the situation is somewhat different. Here the implicit function theorem cannot be applied to all points on the nodal line. However, if we let

$$\Gamma_1 = \text{component of } \{(x, y) \colon y = \inf\{\bar{y} \colon u(x, \bar{y}) = 0\}\}, \quad 0 < \bar{y} < g(x),$$

and define

$$f_1(x, y) = u(x, y) \text{ along } \Gamma_1, \tag{14}$$

then we have the following result.

Theorem 2.4

Let u satisfy problem (1, 2), where the conditions of Theorem 2.3 and

(i) $\phi'(x) < 0$

are satisfied. Then $\forall x \geqslant a > 0$, $f_1(x, y) = 0$ can be solved for x as a function of y. This function is continuously differentiable and monotonically decreasing in y.

Proof. By Theorem 2.3, the solution u changes sign in D. Hence, there exists a boundary curve $u(x, y) = 0$ which separates the negative values of u in D from the positive values of u in a neighborhood of the boundary of D. Let (ξ, η) be a point on the curve $f_1(x, y) = 0$. Integrating equation (1) from 0 to η, we have

$$u_x(x, \eta) = u_x(x, 0) - \int_0^{\eta} p(x, y) u(x, y) \, dy, \quad x \geqslant 0,$$

so that

$$u_x(\xi, \eta) = u_x(\xi, 0) - \int_0^{\eta} p(\xi, \eta) u(\xi, \eta) \, dy. \tag{15}$$

Since $u(\xi, y) > 0$, for $0 < y < \eta$, $u_x(\xi, 0) < 0$ and $p(x, y) > 0$, equation (15) yields $u_x(\xi, \eta) < 0$. Therefore, by the implicit function theorem, $f_1(x, y) = 0$ can be solved uniquely for x as a function of y in a neighborhood of (ξ, η). Since (ξ, η) was an arbitrary point on the graph of $f_1(x, y) = 0$, it follows that $f_1(x, y) = 0$ can be solved for all x for which f_1 is defined. Furthermore, the function $x = h(y)$ obtained in this way is continuously differentiable because u is. Also, from

$$\frac{df_1}{dy} = h'(y) f_{1,x} + f_{1,y} = 0$$

we find that

$$h'(y) = -\frac{f_{1,y}}{f_{1,x}} < 0.$$

Note 1. Theorem 2.4 will still be true if we assume $\phi'(x) \leqslant 0$ and $p(x, y) \geqslant 0$, $p(x, y) \not\equiv 0$.

Note 2. The equation $f_1(x, y) = 0$ can also be solved uniquely for y.

The function $x = h(y)$ of Theorem 2.5 is the closest piece of nodal line to the x-axis. It approaches the x-axis monotonically as $x \to \infty$. The next question is, under what conditions will it become asymptotic to the x-axis? The following result gives an answer to this question.

Theorem 2.5

Let u satisfy problem (1, 2). If

(i) $p(x, y) \geqslant 0$

(ii) $\phi(x), \psi(x) > 0$,

(iii) $\lim_{x \to \infty} \phi(x) = 0$,

(iv) $\phi'(x) \leqslant 0$

and

(v) $\psi(x) < \phi(x)$, for $0 < x \leqslant N$, $N = \text{const}$,

then the curve $f_1(x, y) = 0$ defined by equation (14) is asymptotic to the x-axis.

Proof. Conditions (i)–(v) imply that the curve $f_1(x, y) = 0$ is monotone decreasing by Theorems 2.3 and 2.4. Suppose that on the contrary $f_1(x, y) = 0$ is asymptotic to the line $y = T$, $T > 0$. Let (x, y) be a point in D such that $0 < y < \min[T, g^{-1}(N)]$. The value of u at (x, y) is given by

$$u(x, y) = u(g^{-1}(y), y) - u(g^{-1}(y), 0) + u(x, 0) - \int_{g^{-1}(y)}^{x} \int_0^{y} p(\xi, \eta) u(\xi, \eta) \, d\eta \, d\xi. \tag{16}$$

For such (x, y), u is positive and $u(g^{-1}(y), y) - u(g^{-1}(y), 0)$ is negative, the latter by assumption (v). Holding y fixed and letting x become arbitrarily large the r.h.s. of equation (16) becomes negative, a contradiction. Therefore, $f_1(x, y) = 0$ is asymptotic to $y = 0$.

Note. The result of Theorem 2.5 remains valid if we change the condition $p > 0$, $\phi' \leqslant 0$ to $p \geqslant 0$, $p \not\equiv 0$, $\phi' < 0$.

Another question one would like to ask is whether under specific conditions a sequence of oscillations can exist. If so, what are the locations of other nodal lines compared to the first one? Finally, one would like to know what properties they have. Even though we do not have answers to these questions, we have an example for the easy case $p \equiv 0$.

Example 2.3

The Goursat problem,

$$u_{xy} = 0, \quad x > 0, \quad 0 < y < x,$$

$$u(x, 0) = \frac{2}{x+1}, \quad x \geqslant 0,$$

$$u(x, x) = \frac{2 - \sin x}{x+1}, \quad x \geqslant 0,$$

has the solution

$$u(x, y) = \frac{2}{x+1} - \frac{\sin y}{y+1}.$$

A sketch of nodal lines show that we have a sequence of curves lying in $x > 0$, $0 < y < x$, the first of which is asymptotic to the x-axis. Other nodal lines are asymptotic to the adjacent nodal lines.

Whether a similar behavior can be detected for $p \geqslant 0$, $p \not\equiv 0$, is hard to answer. But we note that the key to the oscillation in Example 2.3 is the fact that $\psi < \phi$ on an infinite sequence of intervals of the positive x-axis.

Finally, we note that Theorems 2.3–2.5 can easily be extended to the nonlinear Goursat problem (5, 6).

3. NONOSCILLATION CRITERIA FOR THE GOURSAT PROBLEM

The nonoscillatory behavior of solutions of problem (1, 2) can be studied by looking at two different cases, namely $p \leqslant 0$ and $p \geqslant 0$. These correspond to a string with repulsive and attractive restoring forces, respectively.

In the case of repulsive forces it is natural to expect that the solution stays positive for positive boundary data and negative for negative boundary data. This intuitive fact is stated as our first result.

Theorem 3.1

Suppose that in problem (1, 2) the following conditions are satisfied:

(i) $p(x, y) \leqslant 0$,

(ii) $\psi(x) \geqslant \phi(x) \geqslant 0$.

Then $u \geqslant 0$ in D.

Proof. Let $R = (x, y)$ be a point in D. Then u at R is given by

$$u(R) = u(T) + u(Q) - u(S) - \int_{g^{-1}(y)}^{x} \int_0^y p(\xi, \eta) u(\xi, \eta) \, d\eta \, d\xi,$$

where T, S and Q have coordinates $(g^{-1}(y), y)$, $(g^{-1}(y), 0)$ and $(x, 0)$, respectively. Choosing $u_0 = u(T) + u(Q) - u(S)$ as the initial value in the iteration

$$u_{n+1} = u(T) + u(Q) - u(S) - \int_{g^{-1}(y)}^{x} \int_0^y p(\xi, \eta) u_n \, d\eta \, d\xi$$

for $u(R)$. It is seen easily that all terms of the sequence $\{u_n\}$ are nonnegative and hence $\{u_n\}$ must converge to a nonnegative function which satisfies problem (1, 2). This yields the desired result because R was chosen arbitrarily.

By the same argument as in Theorem 3.1, and under slightly different conditions, we have the following result.

Theorem 3.2

In problem (1, 2), if

(i) $p(x,y) \leqslant 0$,

(ii) $\phi(x) \geqslant 0, \quad \psi(x) \geqslant 0$,

and

(iii) $\phi'(x) \geqslant 0$,

then $u \geqslant 0$ in D.

Example 3.1

The function $u(x,y) = e^{x+y}$ is the solution of the problem

$$u_{xy} - u = 0,$$
$$u(x,0) = e^x,$$
$$u(x,x) = e^{2x}.$$

Obviously the data satisfies the conditions of Theorems 3.1 and 3.2.

In the case of positive p nonoscillation is less obvious. Physically $p > 0$ corresponds to an attractive force which tends to restore the equilibrium position of the string. Pagan [4] showed that the solution of the characteristic initial-value problem,

$$u_{xy} + p(x,y)u = 0,$$
$$u(x,0) = \phi(x),$$
$$u(0,y) = \psi(y),$$

satisfies the inequality $\psi(y) \leqslant u(x,y) \leqslant \phi(x)$ in $D_1 = \{(x,y)\colon 0 \leqslant x \leqslant X,\ 0 \leqslant y \leqslant Y\}$ if the following conditions hold in D_1:

(i) $p(x,y) \geqslant 0$,

(ii) $\psi(y) > 0, \quad \psi'(y) < 0$

and

(iii) $\phi'(x) \geqslant \phi(x) \displaystyle\int_0^Y p(x,s)\,ds.$

Motivated by this result we obtain the following analogous result for the Goursat problem.

Theorem 3.3

In problem (1, 2), let the following conditions hold in $D' = \{(x,y)\colon 0 \leqslant x \leqslant X,\ 0 \leqslant y \leqslant g(x)\}$:

(i) $p(x,y) > 0$,

(ii) $\phi(x) > 0, \quad \psi(x) > 0$,

(iii) $\phi'(x) \leqslant 0$

and

(iv) $\psi'(x) \geqslant \phi'(x) + g'(x)\psi(x) \displaystyle\int_x^X p(\xi, g(\xi))\,d\xi.$

Then u satisfies

$$\phi(x) \leqslant u(x, y) \leqslant \psi(g^{-1}(y)) \text{ in } D'.$$

Proof. Suppose u is zero somewhere in D. Since u is positive at the origin, there exists a point Q with coordinates (x, y) such that u is positive in the subdomain $D'' = \{(\xi, \eta)$ in $D'\colon g^{-1}(\eta) \leqslant \xi \leqslant x, 0 \leqslant \eta \leqslant y\}$ except at the point (x, y), where $u(x, y) = 0$. From equation (1) and conditions (i) and (iii) we have

$$u_x(\xi, \eta) = \phi'(\xi) - \int_0^\eta p(\xi, t) u(\xi, t)\, \mathrm{d}t < 0, \quad (\xi, \eta) \text{ in } D''. \tag{17}$$

This implies that

$$\psi(g^{-1}(\eta)) = u(g^{-1}(\eta), \eta) \geqslant u(\xi, \eta), \quad (\xi, \eta) \text{ in } D''. \tag{18}$$

Also from equations (1) and (18), we have

$$u_y(\xi, \eta) = u_y(g^{-1}(\eta), \eta) - \int_{g^{-1}(\eta)}^{\xi} p(s, \eta) u(s, \eta)\, \mathrm{d}s$$
$$> u_y(g^{-1}(\eta), \eta) - \int_{g^{-1}(\eta)}^{\xi} p(s, \eta) \psi(g^{-1}(\eta))\, \mathrm{d}s. \tag{19}$$

On the other hand, we have

$$u_y(g^{-1}(y), y) = [g^{-1}(y)]'[\psi'(g^{-1}(y)) - u_x(g)^{-1}(y), y)]. \tag{20}$$

In equation (17), letting $\xi = g^{-1}(\eta)$ we will obtain

$$u_x(g^{-1}(\eta), \eta) = \phi'(g^{-}(\eta)) - \int_0^\eta p(g^{-1}(\eta), t) u(g^{-1}(\eta), t)\, \mathrm{d}t. \tag{21}$$

Equations (20) and (21) and inequality (19) imply that

$$u_y(\xi, \eta) > [g^{-1}(\eta)]'[\psi'(g^{-1}(\eta)) - \phi'(g^{-1}(\eta))$$
$$+ \int_0^\eta p(g^{-1}(\eta), t) u(g^{-1}(\eta), t)\, \mathrm{d}t] - \int_{g^{-1}(\eta)}^{\xi} p(s, \eta) \psi(g^{-1}(\eta))\, \mathrm{d}s. \tag{22}$$

Assumption (iv) implies that

$$[g^{-1}(\eta)]'[\psi'(g^{-1}(\eta)) - \phi'(g^{-1}(\eta))] - \int_{g^{-1}(\eta)}^{\xi} p(s, \eta) \psi(g^{-1}(\eta))\, \mathrm{d}s \geqslant 0.$$

Therefore, by inequality (22),

$$u_y(\xi, \eta) > 0 \text{ in } D''. \tag{23}$$

Now since $u_y(\xi, \eta) > 0$ we must have $\phi(x) < u(x, y)$ in D'', which is a contradiction. Therefore, u is strictly positive in D'. From expressions (17) and (23), we deduce that $\phi(x) \leqslant u(x, y) \leqslant \psi(g^{-1}(y))$ in D'.

Example 3.2

We note that condition (iv) of Theorem 3.3 allows $\psi'(x)$ to be nonpositive. For example, consider the problem

$$\begin{cases} u_{xy} + g'(x) u = 0, \\ u(x, 0) = \phi(x) = \mathrm{e}^{-g(x)}, \\ u(x, g(x)) = \psi(x) = 1, \end{cases}$$

where $g'(x) \geqslant 0$ and $g(0) = 0$, then $u = \mathrm{e}^{y - g(x)}$ satisfies the above problem. For such data, condition (iv) reduces to

$$0 \leqslant g(X) - g(x) \leqslant \mathrm{e}^{-g(x)}. \tag{24}$$

Let

$$\lim_{x\to\infty} g(x) = L. \quad \text{Since } g'(x) \geqslant 0 \text{ for } x \geqslant 0, \ \inf_{x\geqslant 0} g(x) = 0$$

and

$$\sup_{x\geqslant 0} g(x) = L. \quad \text{If } L = L - g(0) \leqslant e^{-L}, \text{ i.e. } Le^{L} \leqslant 1,$$

then inequality (24) is satisfied for all X and x, where $X \geqslant x$. One such function is given by

$$g(x) = \frac{1}{2} - \frac{1}{x+2}.$$

Therefore,

$$u = \exp\left[y - \left(\frac{1}{2} - \frac{1}{x+2}\right)\right]$$

and it satisfies

$$\exp\left[-\left(\frac{1}{2} - \frac{1}{x+2}\right)\right] \leqslant \exp\left[y - \left(\frac{1}{2} - \frac{1}{x+2}\right)\right] \leqslant 1$$

in

$$D = \left\{(x, y)\colon 0 \leqslant x < \infty, \ 0 \leqslant y \leqslant \frac{1}{2} - \frac{1}{x+2}\right\}.$$

The problem of Example 3.2 is important because its solution is strictly positive even though the boundary data is nonincreasing and positive, and $p > 0$. However, for such initial data, the characteristic initial-value problem (P1) must have an oscillatory solution according to Pagan's [4] result.

4. BOUNDEDNESS CRITERIA FOR THE GOURSAT PROBLEM

Krieth and Swanson [7] proved that if

(1) $f(x, y, u) > 0$ for $u > 0$ and monotone in u

and

(2) $$\int^{\infty}\int^{\infty} x^{m-1} y^{n-1} f(x, y, b_0)\,dx\,dy < \infty \text{ for some const } b_0 > 0,$$

then there exists $X \geqslant 0$, $Y \geqslant 0$, such that the equation

$$D_1^m D_2^n u + f(x, y, u) = 0, \quad x > 0, \ y > 0,$$

has a solution which is bounded in $x \geqslant X$, $y \geqslant Y$. Furthermore, they note that if

$$\int^{\infty}\int_0^{X} x^{m-1} y^{n-1} f(x, y, b_0)\,dx\,dy < \infty, \ \forall X > 0,$$

and

$$\int^{\infty}\int_0^{Y} x^{m-l} y^{n-1} f(x, y, b_0)\,dy\,dx < \infty, \ \forall Y > 0,$$

then the bounded solution can be extended to $x \geqslant 0$, $y \geqslant 0$.

This result, as they note, is a generalization of the existence of an asymptotically constant solution of

$$\frac{d^n u}{dx^n} \pm f(x, y) = 0, \quad 0 \leqslant x < \infty,$$

which goes back to Atkinson [11] and Nehari [12] for $n = 2$, with more general ODE results being due to several other authors [13–15]. Generalizations to elliptic equations have been given by Swanson [16, 17] and Kreith and Swanson [7].

This section is an attempt to apply the technique of ref. [7] for establishing the existence of a bounded solution to the equation

$$u_{xy} + f(x, y, u) = 0$$

in the domain $x > g^{-1}(y)$, $y > 0$. Here f is a continuous function of its arguments and g is defined as before. The technique is based on an application of Schauder's fixed-point theorem and a generalization of a result by Levitan [18].

Levitan [18] proved the following. Let S be a closed, bounded, convex subset of a Banach space of continuous functions on $[T, \infty)$. Let ϕ be a continuous map from S into S. Then $\overline{\phi S}$, the closure of the image of ϕ on S, is compact if $[T, \infty)$ can be decomposed into a finite number of subintervals on each of which $\| \phi(y(t_1)) - \phi(y(t_2)) \| < \epsilon$, for y in S and any preassigned number ϵ.

The key point in Ref. [7] is to use Levitan's result when the infinite interval $[T, \infty)$ is replaced by $[X, \infty) \times [Y, \infty)$. In what follows we are going to replace $[T, \infty)$ by $[X, \infty) \times [Y, g(x)]$, x in $[X, \infty)$.

Theorem 4.1

Let $U(x, y) = K(x) + H(y)$ be a positive, uniformly continuous and differentiable function of x and y. Also, let

$$\lim_{x \to \infty} K(x) = M \quad \text{and} \quad \lim_{y \to \infty} H(y) = N,$$

where M and N are const. Assume that

$$\int^{\infty} f(\xi, y, U(\xi, y))\, d\xi < \infty, \ \forall 0 < y < g(x),$$

where $f(x, y, u)$ is continuous in (x, y, u) and a positive, monotone function of u for $x > g^{-1}(y)$, $y > 0$, $u > 0$. Then for every $\epsilon > 0$ there exists Y and a solution u of

$$u_{xy} + f(x, y, u) = 0, \quad x > g^{-1}(y), \quad y > 0,$$

such that $|u(x, y) - U(x, y)| < \epsilon$ for $x > g^{-1}(y)$, $y > Y$.

Proof. Choose Y such that

$$\int_Y^{g(x)} \int_x^{\infty} f(\xi, y, U(\xi, y))\, d\xi\, dy < \epsilon/4 \tag{25}$$

and

$$|H(y)| < \epsilon/8, \quad \text{for } y \geqslant Y.$$

Let $R(x, y) = \{(x, y)\colon x \geqslant g^{-1}(y),\ y \geqslant Y\}$ and B be the space of continuous functions on R with finite sup norm

$$\| u \| = \sup_{R(x, y)} |u(x, y)| < \infty.$$

It can easily be shown that B is a Banach space. Let $S = \{u \text{ in } B\colon \| u - (U + \epsilon/2) \| \leqslant \epsilon/2\}$. Obviously S is bounded. It is also closed and convex. Let the mapping ϕ on S be defined as follows:

$$\phi[u] = U(x, y) + \epsilon/2 - \int_y^{g(x)} \int_x^{\infty} f(\xi, \eta, u(\xi, \eta))\, d\xi\, d\eta. \tag{26}$$

Then, a fixed point of equation (26) is the desired solution.

In order to apply Schauder's fixed-point theorem to equation (26) we need to show

(i) ϕ is continuous on S,

(ii) ϕ maps S into S

and

(iii) $\overline{\phi S}$, the closure of image of the ϕ, is compact.

(i) ϕ *is continuous.* Let $\langle u_n \rangle$ be a sequence of functions in S which converges to u in S, then $f(x, y, u_n(x, y))$ converges to $f(x, y, u(x, y))$. Since f is positive and monotone decreasing, $f(x, y, u_n) \leqslant f(x, y, U)$, $\forall n$. By the monotone convergence theorem, we can change the order of integration and limit in the following:

$$\lim_{n\to\infty} \phi[u_n] = U(x, y) + \frac{\epsilon}{2} - \lim_{n\to\infty} \int_y^{g(x)} \int_x^{\infty} f(\xi, \eta, u_n(\xi, \eta))\, d\xi\, d\eta$$

so that

$$\lim_{n\to\infty} \phi[u_n] = U(x, y) + \frac{\epsilon}{2} - \int_y^{g(x)} \int_x^{\infty} f(\xi, \eta, u(\xi, \eta))\, d\xi\, d\eta = \phi[u].$$

Therefore ϕ is continuous.

(ii) ϕ *maps* S *into* S. From expressions (25) and (26), we have

$$\left\| \phi[u] - \left(U(x, y) + \frac{\epsilon}{2} \right) \right\| = \left\| \int_y^{g(x)} \int_x^{\infty} f(\xi, \eta, u(\xi, \eta))\, d\xi\, d\eta \right\| < \frac{\epsilon}{4}.$$

(iii) $\overline{\phi S}$ *is compact.* Let X be chosen in such a way that $|K(x)| < \epsilon/8$ for $x \geqslant X$. Then for (x_1, y_1), (x_2, y_2) in $\{(x, y)\colon x \geqslant X,\ Y \leqslant y \leqslant g(x)\}$ we have

$$\begin{aligned} \| \phi[u(x_1, y_1)] - \phi[u(x_2, y_2)] \| &\leqslant \| U(x_1, y_1) - U(x_2, y_2) \| + \left\| \int_{y_1}^{g(x_1)} \int_{x_1}^{\infty} f(\xi, \eta, u(\xi, \eta))\, d\xi\, d\eta \right\| \\ &\quad + \left\| \int_{y_2}^{g(x_2)} \int_{x_2}^{\infty} f(\xi, \eta, u(\xi, \eta))\, d\xi\, d\eta \right\| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

For $Y \leqslant y \leqslant g(X)$, $g^{-1}(y) \leqslant x \leqslant X$, we can divide this region into a finite number of subsets, such that on each subset $\| U(x_1, y_1) - U(x_2, y_2) \| < \epsilon/2$, and again conclude that $\| \phi[u(x_1, y_1) - \phi[x_2, y_2)] \| \leqslant \epsilon$. Therefore, by Levitan's result, $\overline{\phi S}$ is compact. If f is monotone increasing, choose

$$S = \left\{ u \text{ in } B\colon \left\| u - \left(U - \frac{\epsilon}{2} \right) \right\| \leqslant \frac{\epsilon}{2} \right\}.$$

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